Factorization of Sturm–Liouville operators: solvable potentials and underlying algebraic structure

2005 J. Phys. A: Math. Gen. 38 371

(http://iopscience.iop.org/0305-4470/38/2/007)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 371–390 doi:10.1088/0305-4470/38/2/007

Factorization of Sturm–Liouville operators: solvable potentials and underlying algebraic structure

M N Hounkonnou[1,2], **K Sodoga**[1,2] **and E S Azatassou**[1,2]

[1] International Chair in Mathematical Physics and Applications (ICMPA), 072 BP 050 Cotonou, Benin

[2] Unité de Recherche en Physique Théorique (URPT), Institut de Mathémathiques et de Sciences Physiques (IMSP), 01 BP 2628 Porto-Novo, Benin

E-mail: norbert_hounkonnou@cipma.net, ksodoga@cipma.net and azatassou2002@yahoo.fr

Received 3 September 2004, in final form 26 October 2004
Published 15 December 2004
Online at stacks.iop.org/JPhysA/38/371

Abstract

In this paper a general method of factorization of Sturm–Liouville (SL) operators is provided. A method to solve SL eigenvalue problems is presented. New classes of exactly solvable potentials are obtained. The supersymmetry and shape invariance approaches are generalized to the SL operators. It is shown that the SL shape invariance potentials have an underlying algebraic structure. This algebra is in general infinite dimensional. The condition of finite algebra is obtained.

PACS numbers: 02.30.Tb, 03.65.Pm

1. Introduction

There has been great interest in the search for exactly solvable physical problems since the early days of quantum mechanics (QM) [1–9]. Exactly solvable here means that the eigenvalues and the eigenfunctions of the Hamiltonian of the system can be derived analytically in closed form. Exactly solvable potentials are important for a number of reasons such as providing model problems to analyse, to start perturbation theory expansions from, or to provide complete sets of basis functions for solving real problems. Because of their major interest, many methods have been developed in order to increase the number of exactly solvable potentials. In this respect, the factorization method plays an important role [10–14]. This technique was first introduced by Schrödinger [13], and later developed by Infeld and Hull [14]. The interest of the subject has been renewed by the introduction of supersymmetry in quantum mechanics (SUSY QM) by Witten [15] and the concept of shape invariance by Gendenshtein [16]. An excellent review of these concepts has been made by Cooper *et al* [17]. It was shown recently that the shape invariance condition [18, 19] has an underlying algebraic structure and the associated Lie algebras were identified [20, 21]. It is worth mentioning that Lévai has developed, with connection to SUSY QM, a nice method of constructing potentials for which the Schrödinger equation can be solved in terms of known special functions [22, 23].

It is remarkable that although the factorization method, SUSY QM and shape invariance concepts were developed to solve the eigenvalue problem related to the time-independent Schrödinger equation, they are also some very powerful tools to solve some classes of second-order differential equations of mathematical physics [24]. It has been shown that the standard differential equations of mathematical physics, e.g., the hypergeometric-type equations, and their associated equations have the properties of SUSY QM and shape invariance; by using these properties, the solutions of these differential equations are expressed in the operator form similar to that for the harmonic oscillator. Indeed, recently, Jafarizadeh and Fakhri [25] have developed a general method of factorization of associated differential equations and obtained their shape invariance relation. Lorente [26] showed later that classical orthogonal polynomial operators can be factorized by using the three-term recurrence relation and a consequence of Rodrigues formulae. Cotfas [27], following Lorente's idea, has obtained a factorization of associated orthogonal polynomial operators, using a change of function relation and a three-term recurrence relation. We have provided, in a previous work [28], a SUSY QM type factorization of the Hamiltonians of the rigidly constrained triatomic molecular systems. All the differential equations mentioned so far, as well as the time-independent Schrödinger equation have a property in common: they fall into the class of (SL) equations [29–31].

The purpose of this paper is to provide a general method of factorization for the SL operators and to extend to these operators Lévai's method of constructing solvable potentials. The motivation for this work is that many problems in quantum mechanics or mathematical physics lead to SL differential equations and the SL operators are well known to be self-adjoint [30]. This is the case of time-independent Schrödinger equations, hypergeometric-type equations, etc. Furthermore, it is often possible to transform a second-order linear differential equation into a SL equation. In addition, it is easy to find an associated potential of Schrödinger type, corresponding to SL potentials, by making coordinate transformations.

This paper is organized as follows. In section 2, we briefly review the usual factorization method and associated algebraic structures. In section 3, we provide a general method of factorization of the SL operators. In section 4, we extend to Sturm–Liouville equations the method given by Lévai to construct solvable potentials for Schrödinger equations. We end with some conclusions in section 5.

2. Usual factorization method and associated algebras: an overview

In this section, we give a brief review of the concepts of the factorization method, SUSY QM, shape invariance and set the notation we shall need in the following. Consider the one-dimensional bound-state Hamiltonian ($\hbar = 2m = 1$)

$$\hat{H} = -D^2 + \tilde{V}(x), \qquad x \in I \subset \mathbb{R}, \tag{1}$$

where $D \equiv \frac{\mathrm{d}}{\mathrm{d}x}$, I is the domain of the variation of x and the potential $\hat{V}(x)$ is a real function, which can have singularities only on the boundary points of I. Let us denote by E_n and $\tilde{\Psi}_n$ the eigenvalues and eigenfunctions of $\hat{H}$, respectively. The factorization method consists of writing Hamiltonian (1) as the product of two first-order mutually adjoint differential operators $\hat{A}$ and $\hat{A}^\dagger$. If the ground-state eigenvalue E_0 and eigenfunction $\tilde{\Psi}_0$ are known, then Hamiltonian (1) factorizes as [8]

$$\hat{H} - E_0 = \hat{A}^\dagger \hat{A} \tag{2}$$

where

$$\hat{A} = D + \tilde{W}(x), \qquad \hat{A}^{\dagger} = -D + \tilde{W}(x) \tag{3}$$

and $\tilde{W}(x) = -D[\ln(\tilde{\Psi}_0)]$.

SUSY QM begins with a set of two matrix operators, known as supercharges

$$\hat{Q}^{+} = \begin{pmatrix} 0 & \hat{A}^{\dagger} \\ 0 & 0 \end{pmatrix}, \qquad \hat{Q}^{-} = \begin{pmatrix} 0 & 0 \\ \hat{A} & 0 \end{pmatrix}. \tag{4}$$

They form the following superalgebra [17],

$$\{\hat{Q}^{+}, \hat{Q}^{-}\} = \hat{H}_{SS}, \qquad [\hat{H}_{SS}, \hat{Q}^{\pm}] = (\hat{Q}^{\pm})^2 = 0, \tag{5}$$

where the SUSY Hamiltonian $\hat{H}_{SS}$ is given by

$$\hat{H}_{SS} = \begin{bmatrix} \hat{A}^{\dagger}\hat{A} & 0 \\ 0 & \hat{A}\hat{A}^{\dagger} \end{bmatrix} = \begin{bmatrix} \hat{H}_1 & 0 \\ 0 & \hat{H}_2 \end{bmatrix}. \tag{6}$$

In terms of the Hermitian supercharges

$$\hat{Q}_1 = (\hat{Q}_+ + \hat{Q}_-)/\sqrt{2} \qquad \text{and} \qquad \hat{Q}_2 = (\hat{Q}_+ - \hat{Q}_-)/\sqrt{2}\mathrm{i},$$

the superalgebra takes the form

$$\{\hat{Q}_i, \hat{Q}_j\} = \hat{H}_{SS}\delta_{ij}, \qquad [\hat{H}_{SS}, \hat{Q}_i] = 0, \qquad i, j = 1, 2, \tag{7}$$

δ_{ij} being the Kronecker symbol. The operators $\hat{H}_1$ and $\hat{H}_2$ given by

$$\hat{H}_1 = \hat{A}^{\dagger}\hat{A} = -D^2 + \tilde{V}_1 = -D^2 + (\tilde{W}^2(x) - \tilde{W}'(x)), \tag{8}$$

$$\hat{H}_2 = \hat{A}\hat{A}^{\dagger} = -D^2 + \tilde{V}_2 = -D^2 + (\tilde{W}^2(x) + \tilde{W}'(x)), \tag{9}$$

are called SUSY partner Hamiltonians; the function $W(x)$ is called the superpotential. The potentials $\tilde{V}_{1,2}$, called SUSY partner potentials, are related to the superpotential by the Riccati-type equations

$$\tilde{V}_1 = \tilde{W}^2(x) - \tilde{W}'(x), \qquad \tilde{V}_2 = \tilde{W}^2(x) + \tilde{W}'(x), \tag{10}$$

where the prime denotes the derivative with respect to x. In terms of $\tilde{W}(x)$ the SUSY Hamiltonian $\hat{H}_{SS}$ reads as

$$\hat{H}_{SS} = (-D^2 + \tilde{W}^2(x))\mathbb{1}_2 + \tilde{W}'(x)\sigma_3, \tag{11}$$

where σ_3 is the Pauli matrix and $\mathbb{1}_2$ is the 2×2 identity matrix.

Let us denote, for $n \geqslant 0$, by $\tilde{\Psi}_n^{(1)}$ and $\tilde{\Psi}_n^{(2)}$ the eigenfunctions of $\hat{H}_1$ and $\hat{H}_2$ with eigenvalues $E_n^{(1)}$ and $E_n^{(2)}$, respectively. It is straightforward to see that the eigenvalues of $\hat{H}_1$ and $\hat{H}_2$ are positive definite $\left(E_n^{1,2} \geqslant 0\right)$ and isospectral, i.e., they have almost the same energy eigenvalues, except for the ground-state energy of $\hat{H}_1$. Their energy spectra are related as [17]

$$\begin{aligned} &E_n = E_n^{(1)} + E_0, \qquad E_0^{(1)} = 0, \qquad \tilde{\Psi}_n = \tilde{\Psi}_n^{(1)}, \qquad n = 0, 1, 2, \ldots, \\ &E_n^{(2)} = E_{n+1}^{(1)}, \qquad E_0^{(1)} = 0, \qquad n = 0, 1, 2, \ldots, \\ &\tilde{\Psi}_n^{(2)} = \left[E_{n+1}^{(1)}\right]^{(-1/2)} \hat{A}\tilde{\Psi}_{n+1}^{(1)}, \\ &\tilde{\Psi}_{n+1}^{(1)} = \left[E_n^{(2)}\right]^{(-1/2)} \hat{A}^{\dagger}\tilde{\Psi}_n^{(2)}. \end{aligned} \tag{12}$$

Hence, if the eigenvalues and eigenfunctions of $\hat{H}_1$ were known, one could immediately derive the eigenvalues and eigenfunctions of $\hat{H}_2$. However, the above relations only give the relationship between the eigenvalues and eigenfunctions of the two partner Hamiltonians, but do not allow us to determine their spectra. A condition of an exact solvability is known as the shape invariance condition; that is, the pair of SUSY partner potentials $\tilde{V}_{1,2}$ is similar in shape and differs only in the parameters that appear in them. Mathematically, the shape invariance condition reads as [16]

$$\tilde{V}_2(x; a_1) = \tilde{V}_1(x; a_2) + \mathcal{R}(a_1), \tag{13}$$

where a_1 is a set of parameters and a_2 is a function of $a_1 (a_2 = f(a_1))$ and $\mathcal{R}(a_1)$ is the non-vanishing remainder independent of x. In such a case, the eigenvalues of $\hat{H}_1$ are given by [16]

$$E_n^{(1)} = \mathcal{R}(a_1) + \mathcal{R}(a_2) + \cdots + \mathcal{R}(a_n), \tag{14}$$

with $a_{k+1} = f^k(a_1)$, i.e., the functions f applied k times. The corresponding unnormalized eigenfunctions are given by [7]

$$\tilde{\Psi}_n(x; a_1) \simeq \prod_{p=1}^{n} \hat{A}^\dagger(x; a_p)\tilde{\Psi}_0(x; a_{n+1}). \tag{15}$$

The shape invariance condition (13) can be rewritten in terms of the factorization operators defined in equations (3),

$$\hat{A}(a_1)\hat{A}^\dagger(a_1) = \hat{A}^\dagger(a_2)\hat{A}(a_2) + \mathcal{R}(a_1), \tag{16}$$

where a_2 is a function of a_1. Here, we consider only the translation class of shape invariance potentials, that is the case where the parameters a_1 and a_2 are related as $a_2 = a_1 + \eta$ [18] and the potentials are known in closed form. The scaling class [19] is not treated here since the potentials, in this case, can only be written as Taylor expansion.

Introducing a similarity transformation T_η that replaces a_1 with a_2 in a given operator [20, 21]

$$\hat{T}_\eta \mathcal{O}(a_1) T_\eta^{-1} = \mathcal{O}(a_1 + \eta) \equiv \mathcal{O}(a_2), \tag{17}$$

and the operators

$$\hat{B}_+ = \hat{A}^\dagger(a_1)\hat{T}_\eta, \qquad \hat{B}_- = \hat{T}_\eta^\dagger \hat{A}(a_1), \tag{18}$$

the Hamiltonian (1) can be factorized in terms of the new operators $\hat{B}_\pm$ as

$$\hat{H} - E_0 = \hat{A}^\dagger(a_1)A(a_1) = \hat{B}_+\hat{B}_-. \tag{19}$$

For the case of translation class of shape invariance potentials, the operator $\hat{T}_\eta$ reads as

$$\hat{T}_\eta = \exp\left(\eta \frac{\partial}{\partial a_1}\right). \tag{20}$$

Using the operator $\hat{B}_+$, the expression (15) of the eigenfunctions takes its simplest form

$$\tilde{\Psi}_n(x; a_1) \simeq \hat{B}_+^n \tilde{\Psi}_0(x; a_1). \tag{21}$$

The operators $B_\pm$ fulfil the commutation relations

$$\begin{aligned} &[\hat{B}_-, \hat{B}_+] = \mathcal{R}(a_0), \qquad [\hat{B}_+, \mathcal{R}(a_0)] = \{\mathcal{R}(a_1) - \mathcal{R}(a_0)\}\hat{B}_+ \\ &[\hat{B}_+, (\mathcal{R}(a_1) - \mathcal{R}(a_0))\, \hat{B}_+] = \{[\mathcal{R}(a_2) - \mathcal{R}(a_1)] - [\mathcal{R}(a_1) - \mathcal{R}(a_0)]\}\hat{B}_+^2, \end{aligned} \tag{22}$$

and so on. In general, the operators in the commutation relations (22) and their Hermitian conjugates form an infinite-dimensional algebra. One can show that those potentials where E_n is given by [21]

$$E_n = \beta n^2 + \delta n + \gamma \tag{23}$$

lead to a finite-dimensional algebra

$$[\hat{B}_+, \hat{B}_-] = \mathcal{R}(a_0), \qquad [\hat{B}_+, \mathcal{R}(a_0)] = 2\beta\hat{B}_+, \qquad [\hat{B}_-, \mathcal{R}(a_0)] = -2\beta\hat{B}_-. \tag{24}$$

Algebra (24) is isomorphic to Heisenberg–Weyl algebra if $\beta = 0$. It is isomorphic to $su(2)$ or $su(1, 1)$, respectively, if $\beta > 0$ or $\beta < 0$. Condition (23) is satisfied when $\mathcal{R}(a_n)$ is linear in a_n. This condition is verified if the superpotential is of the form

$$W(x; a_n) = f(x)a_n + g(x). \tag{25}$$

For $\mathcal{R}(a_n)$ to be linear in a_n and independent of x, the functions $f(x)$ and $g(x)$ must satisfy the equations

$$\begin{aligned} &\eta f'(x) - \eta^2 f^2(x) = \beta, \\ &g'(x) - \eta f(x)g(x) = \frac{\delta}{2} - \beta\frac{a_1}{\eta}, \end{aligned} \tag{26}$$

where $\mathcal{R}(a_n) = \delta + \beta - 2a_1\frac{\beta}{\eta} + 2a_n\frac{\beta}{\eta}$.

3. General method of factorization of SL operators

In this section, we extend, to SL equations, the concepts of factorization method, SUSY QM and shape invariance developed to solve the Schrödinger equation and reviewed in the previous section.

Consider the one-dimensional second-order differential equation:

$$H\Psi = E\Psi, \qquad \Psi, \Psi' \in AC_{\mathrm{loc}}(]a, b[) \tag{27}$$

where

$$H = -\sigma(x)D^2 - \tau(x)D + V(x), \tag{28}$$

E is constant, σ, τ and V are real functions defined on an open interval $]a, b[\subset \mathbb{R}$ and $AC_{\mathrm{loc}}(]a, b[)$ is the set of local absolute continuous functions given by

$$\begin{aligned} &AC_{\mathrm{loc}}(]a, b[) = \{f \in AC[\alpha, \beta], \forall[\alpha, \beta] \subset]a, b[, [\alpha, \beta] \text{ compact}\}, \\ &AC[\alpha, \beta] = \{f \in C[\alpha, \beta], f(x) = f(\alpha) + \int_\alpha^x g(t)\, \mathrm{d}t, g \in L^1[\alpha, \beta]\}. \end{aligned}$$

The suitable Hilbert space is $\mathcal{H} = L^2(]a, b[, \rho(x)\, \mathrm{d}x)$ with the inner product defined by means of a non-negative weight function ρ on $]a, b[$:

$$\langle u, v\rangle = \int_a^b \bar{u}(x)v(x)\rho(x)\, \mathrm{d}x, \qquad \forall u, v \in \mathcal{H}, \tag{29}$$

where $\bar{u}$ is the complex conjugate of u. The domain of the operator H will be examined below. If we choose the weight function ρ such that

$$[\sigma(x)\rho(x)]' = \tau(x)\rho(x), \tag{30}$$

then the differential equation (27) can be reduced to the self-adjoint form [29]:

$$-[\sigma(x)\rho(x)\Psi'(x)]' + [V(x) - E(x)]\rho(x)\Psi(x) = 0, \tag{31}$$

and operator (28) can be written in the equivalent form of SL operator [30]

$$H = \frac{1}{\rho(x)}\left(-\frac{\mathrm{d}}{\mathrm{d}x}p(x)\frac{\mathrm{d}}{\mathrm{d}x} + q(x)\right), \tag{32}$$

where $p(x) = \sigma(x)\rho(x)$ and $q(x) = V(x)\rho(x)$. We require:

(i) $p \in AC_{\text{loc}}(]ab[)$, $p' \in L^2_{\text{loc}}(]a,b[)$, $p^{-1} \in L^1_{\text{loc}}(]a,b[)$ positive and real-valued;
(ii) $q \in L^2_{\text{loc}}(]a,b[)$, real-valued;
(iii) $\rho \in L^1_{\text{loc}}(]a,b[)$, $\rho^{-1} \in L^\infty_{\text{loc}}(]a,b[)$ positive and real-valued.

Equation (31), together with the following boundary condition:

$$\sigma(x)\rho(x)[\bar{u}(x)v'(x) - \bar{u}'(x)v(x)]|_a^b = 0, \qquad \forall\, u, v \in \mathcal{H}, \tag{33}$$

is called a *Sturm–Liouville system* [29]. This boundary condition ensures the self-adjointness of the operator H. Since we want our operator H to be self-adjoint, we take its domain on the Hilbert space $\mathcal{H}$ as

$$\begin{aligned} &\mathcal{D}(H) = \{u \in \mathcal{H}, u, pu' \in AC_{\text{loc}}(]a,b[), Hu \in \mathcal{H}\}, \\ &p(x)[\bar{u}(x)v'(x) - \bar{u}'(x)v(x)]|_a^b = 0, \qquad \forall\, u, v \in \mathcal{D}(H). \end{aligned} \tag{34}$$

It is clear that $\mathcal{D}(H)$ is dense in $\mathcal{H}$ since $C_0^\infty(]a,b[) \subset \mathcal{D}(H)$. One can show that the operator $(H, \mathcal{D}(H))$ is self-adjoint [30].

Remark immediately that, when $\sigma(x)$ and $\tau(x)$ are polynomials of at most second and first degrees, respectively, and $V(x)$ is a constant, we are in the case of hypergeometric-type operators [32]. The case $\sigma = \rho = 1$ can be viewed as the model of one-dimensional particle in the external potential V.

The purpose of this section is to factorize the SL operator $(H, \mathcal{D}(H))$ in terms of two first-order mutually adjoint differential operators. Let the first-order differential operator A be defined by:

$$A = \kappa(x)[D + W(x)], \tag{35}$$

with the domain:

$$\mathcal{D}(A) = \{u \in \mathcal{H}, \kappa u' + \kappa W u \in \mathcal{H}\}, \tag{36}$$

where κ and W are real continuous functions on $]a,b[$. We infer $\mathcal{D}(A)$ dense in $\mathcal{H}$ since $H^{1,2}(]a,b[, \rho(x)\,\mathrm{d}x)$ is dense in $\mathcal{H}$ and $H^{1,2}(]a,b[, \rho(x)\,\mathrm{d}x) \subset \mathcal{D}(H)$, where $H^{m,n}(\Omega)$ is the Sobolev spaces of indices (m,n). We assume that the operator A is closed in $\mathcal{H}$. The adjoint operator $A^\dagger$ of A is given by [30]:

$$\mathcal{D}(A^\dagger) = \{u \in \mathcal{H} | \exists \tilde{v} \in \mathcal{H} : \langle Au, v\rangle = \langle u, \tilde{v}\rangle\, \forall u \in \mathcal{D}(A)\}, \qquad A^\dagger v = \tilde{v}. \tag{37}$$

The explicit expression of $A^\dagger$ is given through the following theorem.

Theorem 3.1. *Suppose the following boundary condition:*

$$\kappa(x)\rho(x)u(x)v(x)|_a^b = 0, \qquad \forall\, u \in \mathcal{D}(A) \quad \textit{and} \quad v \in \mathcal{D}(A^\dagger), \tag{38}$$

is verified. Then the operator $A^\dagger$ can be written as

$$A^\dagger = \kappa(x)\,[-D + W(x) + \alpha(x)] \tag{39}$$

where α is a real continuous function defined by $\alpha(x) \equiv -D\ln[\kappa(x)\rho(x)]$.

Proof. From the definition of the operator A and the inner product (29) we have:

$$\begin{aligned} \langle A\bar{u}, v\rangle &\equiv \int_a^b [\kappa(x)\bar{u}'(x) + \kappa(x)W(x)\bar{u}(x)]v(x)\rho(x)\,\mathrm{d}x \\ &= \kappa(x)\bar{u}(x)v(x)\rho(x)|_a^b - \int_a^b \bar{u}(x)(\kappa(x)v(x)\rho(x))'\,\mathrm{d}x \\ &\quad + \int_a^b \kappa(x)W(x)\bar{u}(x)v(x)\rho(x)\,\mathrm{d}x \end{aligned}$$

$$= \kappa(x)\bar{u}(x)v(x)\rho(x)|_a^b + \int_a^b \bar{u}(x)\kappa(x)\left[-D + W(x) - \frac{\kappa'(x)}{\kappa(x)} - \frac{\rho'(x)}{\rho(x)}\right] v(x)\rho(x)\,\mathrm{d}x$$

$$= \langle \bar{u}, A^\dagger v\rangle \qquad \text{for any} \quad u \in \mathcal{D}(A), \quad v \in \mathcal{D}(A^\dagger). \qquad \square$$

Let H_1 and H_2 be the product operators $A^\dagger A$ and $AA^\dagger$, respectively, with the corresponding domains

$$\begin{aligned} &\mathcal{D}(H_1) = \{u \in \mathcal{D}(A), v = Au \in \mathcal{D}(A^\dagger) \text{ and } A^\dagger v \in \mathcal{H}\}, \\ &\mathcal{D}(H_2) = \{u \in \mathcal{D}(A^\dagger), v = A^\dagger u \in \mathcal{D}(A) \text{ and } Av \in \mathcal{H}\}. \end{aligned} \tag{40}$$

Remark that

$$H^{1,2}\left(]a, b[, \rho(x)\,\mathrm{d}x\right) \subset \mathcal{D}(A) \subset \mathcal{D}(A^\dagger).$$

Then

$$\mathcal{D}(H_1), \mathcal{D}(H_2) \supset H^{2,2}(]a, b[, \rho(x)\,\mathrm{d}x).$$

We infer then that $\mathcal{D}(H_1)$ and $\mathcal{D}(H_2)$ are dense in $\mathcal{H}$. The following theorem gives additional conditions to subject to the functions κ and W so that the operator H factorizes in terms of A and $A^\dagger$.

Theorem 3.2. *Suppose that*

(i) κ and α are related to σ and τ as:

$$\kappa^2 = \sigma; \qquad \kappa(\kappa' - \kappa\alpha) = \tau; \tag{41}$$

(ii) the function W verifies the Riccati-type equation:

$$V - E_0 = \sigma(W^2 - W') - \tau W. \tag{42}$$

Then the operators $H_{1,2}$ are self-adjoint, and:

$$\begin{aligned} &H_1 = A^\dagger A = H - E_0 = -\sigma D^2 - \tau D + \sigma(W^2 - W') - \tau W, \\ &H_2 = AA^\dagger = -\sigma D^2 - \tau D + \sigma(W^2 + W') - (\tau - \sigma')W + \kappa(\kappa\alpha)'. \end{aligned} \tag{43}$$

Proof. The operators $A^\dagger A$ and $AA^\dagger$ are self-adjoint since A and $A^\dagger$ are mutually adjoint and A is closed with $\mathcal{D}(A)$ dense in $\mathcal{H}$. A straightforward computation gives

$$\begin{aligned} &A^\dagger A = -\kappa^2 D^2 - \kappa(\kappa' - \kappa\alpha)D + \kappa^2(W^2 - W') - \kappa(\kappa' - \kappa\alpha)W, \\ &AA^\dagger = -\kappa^2 D^2 - \kappa(\kappa' - \kappa\alpha)D + \kappa^2(W^2 + W') + \kappa(\kappa' + \kappa\alpha)W + \kappa(\kappa\alpha)'. \end{aligned}$$

Equations (43) are readily deduced from the above equations using equations (41) and (42). $\square$

Let us remark that identification (41) is equivalent to relation (30) and the quantity α can also be expressed as $\alpha = \frac{\kappa'}{\kappa} - \frac{\tau}{\sigma}$. We can rewrite the operators $H_{1,2}$ as

$$\begin{aligned} &H_1 = A^\dagger A = -\sigma D^2 - \tau D + V_1, \\ &H_2 = AA^\dagger = -\sigma D^2 - \tau D + V_2, \end{aligned} \tag{44}$$

where

$$\begin{aligned} &V_1 = \sigma(W^2 - W') - \tau W, \\ &V_2 = \sigma(W^2 + W') - (\tau - \sigma')W + \kappa(\kappa\alpha)'. \end{aligned} \tag{45}$$

It clearly appears that the factorization method is extended straightforwardly to Sturm–Liouville operators. The operators A, $A^\dagger$ and H_1, H_2 are equivalent to $\hat{A}$, $\hat{A}^\dagger$ and $\hat{H}_1$, $\hat{H}_2$, respectively, for $\sigma = 1$ and $\tau = 0$. Equations (45) are the Riccati-type equations relating the partner potentials to the superpotential and are equivalent to equations (10) for $\sigma = 1$ and $\tau = 0$. One can construct here also a superalgebra H_{SS} by means of supercharges defined in the same way as in the previous section. Expression (11) of the superalgebra takes here the form

$$\begin{aligned} H_{SS} = &\left[-\sigma(x)D^2 - \tau(x)D + \sigma(x)W^2(x) - \tau(x)W(x) + \tfrac{1}{2}(\sigma'(x)W(x) + (\kappa(x)\alpha(x))')\right]\mathbb{1}_2 \\ &+ \left[\sigma(x)W'(x) + \tfrac{1}{2}(\sigma'(x)W(x) + (\kappa(x)\alpha(x))')\right]\sigma_3. \end{aligned} \tag{46}$$

The SUSY partner Hamiltonians $H_{1,2}$ are here also isospectral. The spectra of H and $H_{1,2}$ are related by equations (12) rewritten for appropriate operators and eigenfunctions. The concept of shape invariance is naturally valid here and almost all the equations related to this concept obtained for Schrödinger equations are applicable to SL equations. Equations (26) become here

$$\begin{aligned} &\sigma(x)(2\eta f'(x) - 2\eta^2 f^2(x)) + \eta f(x)\tau'(x) = 2\beta, \\ &\theta(x) + 2\sigma(x)(g'(x) - 2\eta f(x)g(x)) + \tau'(x)(g(x) - \tfrac{\eta}{2}f(x)) + \tau\eta f(x) = \delta - 2\tfrac{\beta}{\eta}a_1. \end{aligned} \tag{47}$$

In the next section, we will use a simple method developed by Lévai [22] to construct several classes of SL exactly solvable problems with shape invariance potentials.

Let us remark that the SL operator (28) can be related to the Schrödinger-type operator. Indeed, if we make the change of variable $x = x(t)$ such that $\mathrm{d}x/\mathrm{d}t = \kappa(x(t))$ and define the new functions

$$\tilde{\Psi}_n(t) = \sqrt{\kappa(x(t))\rho(x(t))}\Psi_n(x(t)), \tag{48}$$

then equation (27) turns to an equation of the Schrödinger type

$$-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\tilde{\Psi}_n(t) + \tilde{V}\tilde{\Psi}_n(t) = E_n\tilde{\Psi}_n(t), \tag{49}$$

where

$$\tilde{V}(t) = \left[V(x) + \frac{1}{2}\frac{\tau(x)(\kappa(x)\rho(x))' + \kappa^2(x)(\kappa(x)\rho(x))''}{\kappa(x)\rho(x)} - \frac{3}{4}\frac{\kappa^2(x)(\kappa(x)\rho(x))'^2}{(\kappa(x)\rho(x))^2}\right]\Bigg|_{x=x(t)}.$$

The factorization of the associated hypergeometric-type operator [27]

$$H_m = -\sigma(x)D^2 - \tau(x)D + V_m(x), \tag{50}$$

where

$$V_m(x) = \frac{m(m-2)}{4}\frac{\sigma'^2(x)}{\sigma(x)} + \frac{m\tau(x)}{2}\frac{\sigma'(x)}{\sigma(x)} - \frac{1}{2}m(m-2)\sigma''(x) - m\tau'(x), \tag{51}$$

proposed recently by Cotfas [27] is also recovered here. Indeed, taking $W_m = -m\kappa'/\kappa$, one can show that $H_m = A_m^\dagger A_m + \lambda_m$ where $A_m = \kappa[D + W_m]$, $A_m^\dagger = \kappa[-D + W_m + \alpha]$, $\lambda_m = -m/2(m-1)\sigma'' - m\tau'$ and the partner Hamiltonian is $H_m^\dagger = H_{m+1} = A_m A_m^\dagger + \lambda_m$.

The above formalism also includes the interesting case of the Hamiltonians of the form

$$H = -D[G(x)D] + V(x), \qquad G(x) > 0, \quad x \in]a, b[, \tag{52}$$

acting in a Hilbert space $L^2(]a, b[, \mathrm{d}x)$. Indeed, if we take $\sigma(x) = G(x)$ and $\tau(x) = G'(x)$ then $\rho(x) = 1$, $\kappa(x) = \sqrt{G(x)}$ and $\alpha(x) = -1/2D[\ln G(x)]$. Therefore

$$\begin{aligned} H &= -G(x)D^2 - G'(x)D + V(x), \\ H_1 &= -G(x)D^2 - G'(x)D + V_1(x) = A^\dagger A, \\ H_2 &= -G(x)D^2 - G'(x)D + V_2(x) = AA^\dagger, \end{aligned} \tag{53}$$

where

$$A = \sqrt{G(x)}\,[D + W(x)]\,, \qquad A^{\dagger} = \sqrt{G(x)}\,[-D + W(x) - 1/2D(\ln G(x))]\,, \tag{54}$$

and the potentials read as

$$\begin{aligned} V_1(x) &= G(x)[W^2(x) - W'(x) - D(\ln G(x))W(x)], \\ V_2(x) &= G(x)[W^2(x) + W'(x)] - \frac{1}{2}\left(G''(x) - \frac{1}{2}\frac{G'^2(x)}{G(x)}\right). \end{aligned} \tag{55}$$

We end this section by giving two examples which show how to construct a new exactly solvable potential from an old one.

3.1. *Example of Legendre SL-type operator*

Consider the operator

$$H = -(1 - x^2)D^2 + 2xD + V(x), \tag{56}$$

where V is a continuous function on $]-1, 1[$. It is a Sturm–Liouville operator with the parameters of Legendre orthogonal polynomials $\sigma(x) = 1 - x^2, \tau(x) = -2x, \rho(x) = 1$, $x \in]-1, 1[$. This operator is called the associated Legendre operator [33] when $V(x) = \frac{m^2}{1-x^2}, m \in \mathbb{N}$. We shall examine below the case $m = 1$. The factorization gives

$$\begin{aligned} &H - E_0 = H_1 = A^{\dagger}A = -(1 - x^2)D^2 + 2xD + V_1, \\ &H_2 = AA^{\dagger} = -(1 - x^2)D^2 + 2xD + V_2, \\ &A = \sqrt{1 - x^2}(D + W), \qquad A^{\dagger} = \sqrt{1 - x^2}\left(-D + W + \frac{x}{1 - x^2}\right), \\ &V_1 = (1 - x^2)(W^2 - W') + 2xW, \qquad V_2 = (1 - x^2)(W^2 + W') + \frac{1}{1 - x^2}. \end{aligned} \tag{57}$$

In the case of associated Legendre operator ($m = 1$): $V(x) = \frac{1}{1-x^2}$, we obtain:

(i) Superpotential:

$$W(x) = \frac{1 - cx}{(x^2 - 1)(x - c)}, \qquad |c| > 1. \tag{58}$$

(ii) Partner potentials:

$$V_1(x) = \frac{1}{1 - x^2}, \qquad V_2(x) = 2\frac{1 - cx}{(c - x)^2}, \quad |c| > 1. \tag{59}$$

(iii) Eigenvalues and eigenfunctions of H_1 [33]:

$$E_n^{(1)} = n(n + 1), \qquad \psi_n^{(1)} = P_n^1(x), \tag{60}$$

where the $P_n^m, m \geqslant n$ are the associated Legendre polynomials [33].

(iv) Eigenvalues and eigenfunctions of H_2:
We deduce the eigenvalues and eigenfunctions of H_2 from those of H_1 using the relations (12):

$$\begin{aligned} &E_n^{(2)} = (n + 1)(n + 2), \qquad n = 0, 1, 2, \ldots \\ &\psi_n^{(2)} = \frac{1}{\sqrt{(n + 1)(n + 2)}}\frac{1}{(x - c)\sqrt{1 - x^2}}\big[(c - x)(n + 2)P_n^1(x) \\ &\qquad + \big(1 + x(x + nx - c(n + 2))P_{n+1}^1(x)\big)\big], \qquad n = 0, 1, 2, \ldots. \end{aligned} \tag{61}$$

3.2. Example of Laguerre SL-type operator

We consider here the operator

$$H = -\frac{\mathrm{d}}{\mathrm{d}x}\left[x\frac{\mathrm{d}}{\mathrm{d}x}\right] + V(x), \qquad x \in]0, +\infty[, \tag{62}$$

where V is a continuous function on $]0, +\infty[$. It is a Sturm–Liouville operator with the parameters of Laguerre orthogonal polynomials $\sigma(x) = x$, $\tau(x) = 1$, $\rho(x) = 1$, $x \in]0, \infty[$. This operator is called Laguerre operator of order $m \in \mathbb{N}$ [31], when $V(x) = V_m(x) = m - 1/2 + x/4 + m^2/4x$. We shall examine below the case $m = 1$ in V_m. The factorization gives

$$\begin{aligned}
&H - E_0 = H_1 = A^\dagger A = -xD^2 - D + V_1,\\
&H_2 = AA^\dagger = -xD^2 - D + V_2,\\
&A = \sqrt{x}(D + W), \qquad A^\dagger = \sqrt{x}\left(-D + W - \frac{1}{2x}\right),\\
&V_1 = x(W^2 - W') - W, \qquad V_2 = x(W^2 + W') + \frac{1}{4x}.
\end{aligned} \tag{63}$$

For the Laguerre operator of order $m = 1$, $V(x) = \frac{x}{4} + \frac{1}{4x}$ and we obtain:

(i) Superpotential :

$$W(x) = -\frac{1 + x + \exp(x + c)(x - 1)}{2x(\exp(x + c) - 1)}, \qquad c > 0. \tag{64}$$

(ii) Partner potentials:

$$V_1(x) = \frac{x}{4} + \frac{1}{4x}, \qquad V_2(x) = \frac{x(3 + \cosh(c + x)) - 2\sinh(c + x)}{4(\cosh(c + x) - 1)}, \qquad c > 0. \tag{65}$$

(iii) Eigenvalues and eigenfunctions of H_1 [31]:

$$E_n^{(1)} = n + 1, \qquad \psi_n^{(1)} = x^{\frac{1}{2}} \exp\left(-\frac{x}{2}\right) L_n^1(x), \tag{66}$$

where the $L_n^m, m \geqslant n$ are the generalized Laguerre polynomials [33].

(iv) Eigenvalues and eigenfunctions of H_2:
We deduce the eigenvalues and eigenfunctions of H_2 from those of H_1 using the relations (12):

$$\begin{aligned}
&E_n^{(2)} = n + 2, \quad n = 0, 1, 2, \ldots,\\
&\psi_n^{(2)} = \frac{1}{\sqrt{n+2}} \frac{\exp\left(-\frac{x}{2}\right)}{\exp(c + x) - 1}((1 - \exp(c + x))xL_n^2(x)\\
&\qquad - (1 + \exp(c + x)(x - 1))L_{n+1}^1(x)), \qquad n = 0, 1, 2, \ldots.
\end{aligned} \tag{67}$$

Let us note that the $\psi_n^{(1)}$ fulfil the following recurrence relation,

$$(n + 1)\psi_{n+1}^{(1)}(x) - (2n + 2 - x)\psi_n^{(1)}(x) + (n + 1)\psi_{n-1}^{(1)}(x) = 0, \tag{68}$$

which can be straightforwardly deduced from the recurrence relation satisfied by the generalized Laguerre functions $L_n^\alpha(x)$ [32]

$$(n + 1)L_{n+1}^\alpha(x) - (2n + \alpha + 1 - x)L_n^\alpha(x) + (n + \alpha)L_{n-1}^\alpha(x) = 0. \tag{69}$$

4. Construction of solvable potentials

Lévai, in a nice paper [22], has developed an elegant method, related to SUSY QM, of constructing solvable potentials for which the Schrödinger equation can be solved exactly in terms of special functions. In this section, we use the same method to construct new exactly solvable potentials for SL equations. Let us start with the SL eigenvalue problem

$$-\sigma(x)\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} - \tau(x)\frac{\mathrm{d}\psi}{\mathrm{d}x} + (V - E)\psi(x) = 0, \tag{70}$$

characterized by the coefficients σ and τ and the potential V. We search for potentials for which the eigenfunctions ψ are expressed as:

$$\psi(x) = f(x)F(y(x)), \tag{71}$$

where F is a special function of the variable y satisfying the equation

$$\frac{\mathrm{d}^2F}{\mathrm{d}y^2} + Q(y)\frac{\mathrm{d}F}{\mathrm{d}y} + R(y)F(y) = 0. \tag{72}$$

Inserting (71) in equation (70) and after a straightforward computation, we obtain

$$\frac{\mathrm{d}^2F}{\mathrm{d}y^2} + \left[\frac{y''}{y'^2} + 2\frac{f'}{fy'} + \frac{\tau}{\sigma y'}\right]\frac{\mathrm{d}F}{\mathrm{d}y} + \left[\frac{E-V}{\sigma y'^2} + \frac{f''}{fy'^2} + \frac{\tau}{\sigma}\frac{f'}{fy'^2}\right]F(y) = 0. \tag{73}$$

Identifying (73) with (72), we deduce

$$\left[\frac{y''}{y'^2} + 2\frac{f'}{fy'} + \frac{\tau}{\sigma y'}\right] = Q(y), \tag{74}$$

$$V - E = -R(y)\sigma y'^2 + \sigma\frac{f''}{f} + \tau\frac{f'}{f}. \tag{75}$$

Observing that $\frac{f''}{f} = \left(\frac{f'}{f}\right)^2 + \left(\frac{f'}{f}\right)'$, equation (75) can be rewritten as:

$$V - E = -R(y)\sigma y'^2 + \sigma\left(\left(\frac{f'}{f}\right)^2 + \left(\frac{f'}{f}\right)'\right) + \tau\frac{f'}{f}, \tag{76}$$

$$= -R(y)\sigma y'^2 + \sigma(W^2 - W') - \tau W, \tag{77}$$

where $W = -f'/f = -(\ln f)'$. From (77), it appears that this method is closely related to the theory of SUSY QM factorization. Indeed, if the function R depends on an integer n such that it vanishes for $n = 0$, that is $R(y) \equiv R(y; n) = n\gamma(n; y)$, then we obtain the Riccati-type equation (42)

$$V - E_0 = \sigma(W^2 - W') - \tau W. \tag{78}$$

W is the superpotential and f plays the role of ground-state eigenfunction. From (74) we derive

$$\frac{f'}{f} = \frac{1}{2}\left(y'Q(y) - \frac{y''}{y'} - \frac{\tau}{\sigma}\right) \tag{79}$$

and the function f reads as

$$f \simeq (y')^{-1/2}\exp\left(\frac{1}{2}\int^{y(x)} Q(t)\,\mathrm{d}t\right)\exp\left(-\frac{1}{2}\int^{x}\frac{\tau(t)}{\sigma(t)}\,\mathrm{d}t\right). \tag{80}$$

Expression (80) differs from the corresponding one found by Lévai by the *extrafunction* term

$$f_{\mathrm{ex}} = \exp\left(-\frac{1}{2}\int^{x}\frac{\tau(t)}{\sigma(t)}\,\mathrm{d}t\right). \tag{81}$$

Introducing (79) in (76), we obtain the expression

$$V - E = \sigma \left\{ -\frac{1}{4}\frac{\tau^2}{\sigma^2} - \frac{1}{2}\left(\frac{\tau}{\sigma}\right)' \right\}$$
$$+\sigma \left\{ -\frac{1}{2}\frac{y'''}{y'} + \frac{3}{4}\left(\frac{y''}{y'}\right)^2 + y'^2\left(-R(y) + \frac{1}{2}\frac{\mathrm{d}Q}{\mathrm{d}y} + \frac{1}{4}Q^2(y)\right)\right\}. \tag{82}$$

Apart from the coefficient σ, expression (82) differs from the corresponding Lévai relation by the *extrapotential* term

$$V_{\mathrm{ex}} = \sigma \left\{ -\frac{1}{4}\frac{\tau^2}{\sigma^2} - \frac{1}{2}\left(\frac{\tau}{\sigma}\right)' \right\}. \tag{83}$$

Let us note that we can make coordinate transformation such that the coefficient σ becomes $\sigma = 1$. Without loss of generality, we set in the following $\sigma = 1$. The expressions of $V - E$ and f become, respectively,

$$V - E = V_{\mathrm{ex}} - \frac{1}{2}\frac{y'''}{y'} + \frac{3}{4}\left(\frac{y''}{y'}\right)^2 + y'^2\left(-R(y) + \frac{1}{2}\frac{\mathrm{d}Q}{\mathrm{d}y} + \frac{1}{4}Q^2(y)\right), \tag{84}$$

$$f \simeq (y')^{-1/2} \exp\left(\frac{1}{2}\int^{y(x)} Q(t)\,\mathrm{d}t\right) f_{\mathrm{ex}}, \tag{85}$$

where $V_{\mathrm{ex}} = -\frac{1}{4}\tau^2 - \frac{1}{2}\tau'$ and $f_{\mathrm{ex}} = \exp\left(-\frac{1}{2}\int^x \tau(t)\,\mathrm{d}t\right)$.

Let us now apply this formalism to Hermite H_n, generalized Laguerre L_n^α and Jacobi $P_n^{(\alpha,\beta)}$ classical orthogonal polynomials. For these special functions, the coefficients $R(y; n)$ vanish for $n = 0$.

4.1. Exactly solvable potentials for Hermite orthogonal polynomials

For Hermite polynomials, the coefficients $Q(y)$ and $R(y)$ are

$$Q(y) = -2y, \qquad R(y) = 2n. \tag{86}$$

Then

$$V - E = V_{\mathrm{ex}} + \left\{ -\frac{1}{2}\frac{y'''}{y'} + \frac{3}{4}\left(\frac{y''}{y'}\right)^2 - y'^2(1 + 2n) - y'^2y^2 \right\}, \tag{87}$$

$$f \simeq (y')^{-1/2} \exp\left(-\frac{y^2}{2}\right) f_{\mathrm{ex}}. \tag{88}$$

One term at least of the right-hand side of (87) must be constant to play the role of constant energy E of the left-hand side. This constant term must contain the integer n. This condition is achieved if we set $y'^2 = \text{constant}$ or $y'^2y^2 = \text{constant}$. $y'^2 = \text{constant}$ implies that $y(x)$ must be linear in x. Using, here and in the following, the parameters of Dabrowska *et al* [18] or Lévai [22], we have $y(x) = \left(\frac{\omega}{2}\right)^2\left(x - \frac{2b}{\omega}\right)$. A possible solution of $y'^2y^2 = \text{constant}$ is $y(x) = 2C^{1/2}x^{1/2}$.

4.2. Exactly solvable potentials for generalized Laguerre orthogonal polynomials

Here the coefficients $Q(y)$ and $R(y)$ are given by

$$Q(y) = \frac{\alpha + 1}{y} - 1, \qquad R(y) = \frac{n}{y}. \tag{89}$$

Then

$$V - E = V_{\rm ex} - \frac{1}{2}\frac{y'''}{y'} + \frac{3}{4}\left(\frac{y''}{y'}\right)^2 - \frac{y'^2}{y}\left(\frac{1}{2}(\alpha+1) + n\right) + \frac{y'^2}{4} + \frac{1}{4}\frac{y'^2}{y^2}(\alpha^2 - 1), \tag{90}$$

$$f \simeq (y')^{-1/2} y^{(\alpha+1)/2} \exp\left(-\frac{y}{2}\right) f_{\rm ex}. \tag{91}$$

To have a constant term on the right-hand side of (90) we can set $\frac{y'^2}{y}$ = constant, y'^2 = constant or $\frac{y'^2}{y^2}$ = constant. These differential equations give, respectively, the following solutions $y(x) = \frac{1}{2}\omega x^2$, $y(x) = \frac{e^2 x}{(n+l+1)}$ and $y(x) = \frac{2B}{a}\exp(-ax)$.

4.3. Exactly solvable potentials for Jacobi orthogonal polynomials

The coefficients $Q(y)$ and $R(y)$ are given by

$$Q(y) = \frac{\beta - \alpha}{1 - y^2} - (\alpha + \beta + 2)\frac{y}{1 - y^2}, \qquad R(y) = \frac{1}{1 - y^2} n(n + \alpha + \beta + 1). \tag{92}$$

Then

$$\begin{aligned} V - E = V_{\rm ex} &- \frac{1}{2}\frac{y'''}{y'} - \frac{3}{4}\left(\frac{y''}{y'}\right)^2 - \frac{y'^2}{1 - y^2} n(n + \alpha + \beta + 1) - \frac{y'^2}{(1 - y^2)^2} \\ &\times \left(\frac{1}{2}(\alpha + \beta + 2) - \frac{1}{4}(\beta - \alpha)^2\right) - \frac{y'^2 y}{(1 - y^2)^2}\left(\frac{1}{2}(\beta - \alpha)(\beta + \alpha)\right) \\ &- \frac{y'^2 y^2}{(1 - y^2)^2}\left(\frac{1}{4} - \left(\frac{\alpha + \beta + 1}{2}\right)^2\right) \end{aligned} \tag{93}$$

$$f \simeq (y')^{-1/2}(1 + y)^{\frac{\beta+1}{2}}(1 - y)^{\frac{\alpha+1}{2}} f_{\rm ex}. \tag{94}$$

To have a constant term on the right-hand side of (93) we can set $\frac{y'^2}{1-y^2}$ = constant or $\frac{y'^2}{(1-y^2)^2}$ = constant. These differential equations give, respectively, the following solutions $y(x) = \{\mathrm{i}\sinh(ax), \cosh(ax), \cosh(2ax), \cos(ax), \cos(2ax)\}$ and $y(x) = \{\tanh(ax), \coth(ax), -\mathrm{i}\cot(ax)\}$.

For the next, we shall investigate the cases of specific SL problems characterized by the coefficients σ, τ and the weight function ρ. Specific SL operators will differ each from other by the extrapotential and extrafunction terms $V_{\rm ex}$ and $f_{\rm ex}$.

4.4. Expressions of $V_{\rm ex}$ and $f_{\rm ex}$ for some particular SL problems

(i) *Hermite SL-type operator*
It corresponds to equation (70) with $\sigma(x) = 1, \tau(x) = -2x$; the Hilbert space is $L^2(\mathbb{R}, \exp(-x^2)\,\mathrm{d}x)$. For these parameters, we have

$$V_{\rm ex} = 1 - x^2, \qquad f_{\rm ex} = \exp\left(\frac{x^2}{2}\right). \tag{95}$$

(ii) *Laguerre SL-type operator*
In this case, $\sigma(x) = x$, $\tau(x) = \mu + 1 - x$, and the Hilbert space is $L^2(\mathbb{R}_+, x^\mu \exp(-x)\,\mathrm{d}x)$. By making the change of variable $x = r^2/4$, the coefficients become $\sigma(r) = 1$,

$\tau(r) = \frac{2\mu+1}{r} - \frac{r}{2}$. The Hilbert space becomes $L^2(\mathbb{R}_+, r^{2\mu+1}\exp(-r^2/4)\,\mathrm{d}r)$. These parameters lead to

$$f_{\mathrm{ex}} = r^{-\mu-1/2}\exp(r^2/8), \qquad V_{\mathrm{ex}} = \left(\frac{1}{4} - \mu^2\right)\frac{1}{r^2} + \frac{1}{2}(\mu+1) - \frac{r^2}{16}. \tag{96}$$

(iii) *Legendre SL-type operator*

Equation (70) rewrites with $\sigma(x) = 1 - x^2$, $\tau(x) = -2x$, and the Hilbert space is $L^2(]-1, 1[, \mathrm{d}x)$. By making the change of variable $x = \cos\theta$, we arrive at $\sigma(\theta) = 1$, $\tau(\theta) = \cot\theta$ with the Hilbert space $L^2(]0, \pi[, \sin\theta\,\mathrm{d}\theta)$. We then obtain

$$f_{\mathrm{ex}} = \sin\theta^{-1/2}, \qquad V_{\mathrm{ex}} = -\tfrac{1}{4}\cot^2\theta - \tfrac{1}{2}\csc^2\theta. \tag{97}$$

(iv) *Chebyshev SL-type operator*

$\sigma(x) = 1 - x^2$, $\tau(x) = -x$; the Hilbert space is $L^2(]-1, 1[, 1/\sqrt{1-x^2}\,\mathrm{d}x)$. By making the change of variable $x = \sin\theta$, we get $\sigma(\theta) = 1$, $\tau(\theta) = 0$ corresponding to the Hilbert space $L^2(]0, \pi[, \mathrm{d}\theta)$. Thus, we have

$$f_{\mathrm{ex}} = 1, \qquad V_{\mathrm{ex}} = 0. \tag{98}$$

In tables 1, 3 and 5 we give the results of computation of exactly solvable potentials for Hermite, Laguerre and Legendre SL problems, respectively. These potentials are similar to the well-known exactly solvable potentials for Schrödinger equation. The only difference is due to the presence of the extrapotential term. As for the Schrödinger case, these potentials are shape invariant. Data on the superpotentials, the shape invariance parameters a_1, a_2 and $\mathcal{R}(a_1)$ are collected in tables 2, 4 and 6, while the corresponding shape-invariance algebras are grouped in table 7. We can see that the shape invariance algebras of Coulomb (LII), Rosen–Morse (PII), Eckart (PII) and Lévai (PII) potentials are infinite-dimensional; whereas those of shifted oscillator (HI), three-dimensional oscillator (LI), Morse (LIII), Eckart (PI), Pöschl–Teller I (PI) and Pöschl–Teller II (PI) potentials are finite.

The eigenvalues of the potentials in the tables are shifted by constant terms so that the first eigenvalues vanish. The potentials are then those of the first partner potentials V_1. The potentials V are different from V_1 by the constant terms. The eigenvalues E_n are different from the eigenvalues $E_n^{(1)}$ by the same constant terms. For example, the Coulomb potential of Laguerre SL-type operator in table 3 gives

$$\begin{aligned}
V_1(r) &= -\frac{e^2}{r} + \frac{l(l+1)}{r^2} + \frac{e^4}{4(l+1)^2} + \left(\frac{1}{4} - \mu^2\right)\frac{1}{r^2} + \frac{1}{2}(\mu+1) - \frac{r^2}{16}, \\
E_n^{(1)} &= \frac{e^4}{4(l+1)^2} - \frac{e^4}{4(n+l+1)^2}, \\
\psi_n^{(1)} &= r^{-\mu-\frac{1}{2}}\exp\left\{\frac{r^2}{8} - \frac{y(r;l)}{2}\right\} y(r;l)^{l+1} L_n^{2l+1}(y(r;l)), \qquad y(r;l) = \frac{re^2}{(n+l+1)}, \\
V(r) &= -\frac{e^2}{r} + \frac{l(l+1)}{r^2} + \left(\frac{1}{4} - \mu^2\right)\frac{1}{r^2} + \frac{1}{2}(\mu+1) - \frac{r^2}{16}, \\
E_n &= -\frac{e^4}{4(n+l+1)^2}, \qquad \psi_n = \psi_n^{(1)}.
\end{aligned} \tag{99}$$

Table 1. Solvable potentials associated with Hermite SL-type problem. We use the notation of Dabrowska *et al* [18] and Lévai [22]. The ranges of the potentials are $-\infty < x < +\infty$, $0 < r < \infty$, $0 < a\theta < \pi$ and $0 < 2a\delta < \pi$. ($A = sa$, $B = \lambda a$ for the case PI and LII and $B = \lambda a^2$ for case PIII.) The data are those of the partner potentials V_1. Those of V_2 can be straightforwardly deduced from (12).

Type of potentials	Variable y	Potential V_1	Eigenvalues $E_n^{(1)}$	Eigenfunctions $\psi_n^{(1)}$
Shifted oscillator HI	$\left(\frac{1}{2}\omega\right)^{\frac{1}{2}}\left(x-\frac{2b}{\omega}\right)$	$\frac{\omega^2}{4}\left(x-\frac{2b}{\omega}\right)^2+1-x^2-\frac{\omega}{2}$	$n\omega$	$\mathrm{e}^{(\frac{x^2}{2}-\frac{y^2}{2})}H_n(y)$
Three-dimensional oscillator LI	$\frac{\omega r^2}{2}$	$\frac{\omega^2 r^2}{4}+\frac{l(l+1)}{r^2}+1-r^2-(l+\frac{3}{2})\omega$	$2n\omega$	$\mathrm{e}^{(\frac{r^2}{2}-\frac{y}{2})}y^{(l+1)/2}L_n^{l+1/2}(y)$
Coulomb LII	$\frac{re^2}{(n+l+1)}$	$\frac{l(l+1)}{r^2}-\frac{e^2}{r}+1-r^2+\frac{e^4}{4(l+1)^2}$	$\frac{e^4}{4(l+1)^2}-\frac{e^4}{4(n+l+1)^2}$	$\mathrm{e}^{(\frac{r^2}{2}-\frac{y}{2})}y^{l+1}L_n^{2l+1}(y)$
Morse LIII	$\frac{2B}{a}\,\mathrm{e}^{-ax}$	$B^2\,\mathrm{e}^{-2ax}-B(2A+a)\,\mathrm{e}^{-ax}$ $+A^2+1-x^2$	$A^2-(A-na)^2$	$\mathrm{e}^{(\frac{x^2}{2}-\frac{y}{2})}y^{s-n}L_n^{2s-2n}(y)$
Morse PI	$\mathrm{i}\sinh(ax)$	$(B^2-A^2-Aa)\,\mathrm{sech}^2(ax)$ $+B(2A+a)\,\mathrm{sech}(ax)\tanh(ax)$ $+A^2+1-x^2$	$A^2-(A-na)^2$	$\mathrm{e}^{\frac{x^2}{2}-2\lambda\arctan(\tanh(\frac{ax}{2}))}$ $\times\cosh^{-s}(ax)$ $\times P_n^{(-\mathrm{i}\lambda-s-\frac{1}{2},\mathrm{i}\lambda-s-\frac{1}{2})}(y)$
Rosen–Morse PII	$\tanh(ax)$ $\bar{a}=\frac{\lambda}{(s-n)}$	$-A(A+a)\,\mathrm{sech}^2(ax)$ $+2B\tanh ax$ $+A^2+\frac{B^2}{A^2}+1-x^2$	$A^2-(A-na)^2$ $+\frac{B^2}{A^2}-\frac{B^2}{(A-na)^2}$	$\mathrm{e}^{(\frac{x^2}{2}-\frac{Bx}{A-an})}$ $\times\cosh^{n-s}(ax)$ $\times P_n^{(s-n+\bar{a},s-n-\bar{a})}(y)$
Rosen–Morse PI	$\cosh(ar)$	$(B^2+A^2+Aa)\,\mathrm{cosech}^2(ar)$ $-B(2A+a)\coth(ar)$ $\times\mathrm{cosech}(ar)+A^2+1-r^2$	$A^2-(A-na)^2$	$\mathrm{e}^{\frac{r^2}{2}}\cosh^{-\lambda-s}\left(\frac{ar}{2}\right)$ $\times\sinh^{\lambda-s}\left(\frac{ar}{2}\right)$ $\times P_n^{(\lambda-s-\frac{1}{2},-\lambda-s-\frac{1}{2})}(y)$
Eckart PII	$\coth(ar)$	$A(A-a)\,\mathrm{cosech}^2(ar)$ $-2B\coth ar$ $+A^2+\frac{B^2}{A^2}+1-r^2$	$A^2-(A+na)^2$ $+\frac{B^2}{A^2}-\frac{B^2}{(A+na)^2}$	$\mathrm{e}^{\frac{r^2}{2}-\frac{Br}{A+an}}$ $\times\sinh^{s+n}(ar)$ $\times P_n^{(-s-n+\bar{a},-s-n-\bar{a})}(y)$
Eckart PI	$\cos(a\theta)$	$(B^2+A^2-Aa)\csc^2(a\theta)$ $-B(2A-a)\cot(a\theta)\csc(a\theta)$ $+1-\theta^2-A^2$	$(A+na)^2-A^2$	$\mathrm{e}^{\frac{\theta^2}{2}}\cos^{s+\lambda}\left(\frac{a\theta}{2}\right)$ $\times\sin^{s-\lambda}\left(\frac{a\theta}{2}\right)$ $\times P_n^{(s-\lambda-\frac{1}{2},s+\lambda-\frac{1}{2})}(y)$
Poschl–Teller I PI	$\cos(2a\delta)$	$A(A-a)\sec^2(a\delta)$ $+B(B-a)\csc^2(a\delta)$ $+1-\delta^2-(A+B)^2$	$(A+B+2na)^2$ $-(A+B)^2$	$\mathrm{e}^{\frac{\delta^2}{2}}\cos^{s}(a\delta)$ $\times\sin^{\lambda}(a\delta)$ $\times P_n^{(\lambda-\frac{1}{2},s-\frac{1}{2})}(y)$
Poschl–Teller II PI	$\cosh(2ar)$	$-A(A+a)\,\mathrm{sech}^2(ar)$ $+B(B-a)\,\mathrm{cosech}^2(ar)$ $+1-r^2+(A-B)^2$	$-(A-B-2na)^2$ $+(A-B)^2$	$\mathrm{e}^{\frac{r^2}{2}}\cosh^{-s}(ar)$ $\times\sinh^{\lambda}(ar)$ $\times P_n^{(\lambda-\frac{1}{2},-s-\frac{1}{2})}(y)$
Lévai PII	$-\mathrm{i}\cot(a\theta)$ $\bar{a}=\frac{\lambda}{(s-n)}$	$A(A+a)\csc^2(a\theta)$ $-2B\cot(a\theta)$ $-A^2+\frac{B^2}{A^2}+1-\theta^2$	$-A^2+(A-na)^2$ $+\frac{B^2}{A^2}-\frac{B^2}{(A-na)^2}$	$\mathrm{e}^{(\frac{\theta^2}{2}+\frac{B\theta}{A-an})}$ $\times\sin^{-s+n}(a\theta)$ $\times P_n^{(s-n+\mathrm{i}\bar{a},s-n-\mathrm{i}\bar{a})}(y)$

The explicit expressions of the eigenfunctions and eigenvalues of the second partner potentials V_2 can be straightforwardly deduced from (12). We do not put the results on V_2 potentials in the tables as the expressions are too cumbersome and long. For the above Laguerre case, we

Table 2. Shape invariance data on solvable potentials for Hermite SL-type problem.

Type of potentials	Variable y	Superpotential W	a_1	a_2	$\mathcal{R}(a_1)$
Shifted oscillator HI	$\left(\frac{\omega}{2}\right)^{\frac{1}{2}}\left(x-\frac{2b}{\omega}\right)$	$\frac{\omega x}{2}-x-b$	ω	ω	ω
Three-dimensional oscillator LI	$\frac{1}{2}\omega r^2$	$\frac{\omega r}{2}-r-\frac{(l+1)}{r}$	l	$l+1$	2ω
Coulomb LII	$\frac{re^2}{(n+l+1)}$	$\frac{e^2}{2(l+1)}-r-\frac{(l+1)}{r}$	l	$l+1$	$\frac{e^4}{4(l+1)^2}-\frac{e^4}{4(l+2)^2}$
Morse LIII	$\frac{2B}{a}\,\mathrm{e}^{(-ax)}$	$A-x-B\,\mathrm{e}^{(-ax)}$	A	$A-a$	$A^2-(A-a)^2$
Morse PI	$\mathrm{i}\sinh(ax)$	$A\tanh(ax)$ $+B\,\mathrm{sech}(ax)-x$	A	$A-a$	$A^2-(A-a)^2$
Rosen–Morse PII	$\tanh(ax)$	$A\tanh(ax)-x+\frac{B}{A}$	A	$A-a$	$A^2-(A-a)^2$ $+\frac{B^2}{A^2}-\frac{B^2}{(A-a)^2}$
Rosen–Morse PI	$\cosh(ar)$	$A\coth(ar)-r-B\,\mathrm{cosech}(ar)$	A	$A-a$	$A^2-(A-a)^2$
Eckart PII	$\coth(ax)$	$\frac{B}{A}-A\coth(ar)-r$	A	$A+a$	$A^2-(A+a)^2$ $+\frac{B^2}{A^2}-\frac{B^2}{(A-a)^2}$
Eckart PI	$\cos(a\theta)$	$B\csc(a\theta)-A\cot(a\theta)-\theta$	A	$A+a$	$(A+a)^2-A^2$
Pöschl–Teller I PI	$\cos(2a\delta)$	$A\tan(a\delta)-\delta-B\cot(a\delta)$	(A,B)	$(A+a,B+a)$	$-(A+B)^2$ $+(A+B+2a)^2$
Pöschl–Teller II PI	$\cosh(2ar)$	$A\tanh(ar)-r-B\coth(ar)$	(A,B)	$(A-a,B+a)$	$(A-B)^2$ $-(A-B-2a)^2$
Lévai PII	$-\mathrm{i}\cot(a\theta)$	$-\frac{B}{A}-\theta-A\cot(a\theta)$	A	$A-a$	$-A^2+(A-a)^2$ $+\frac{B^2}{A^2}-\frac{B^2}{(A-a)^2}$

have

$$
\begin{aligned}
V_2(r) &= -\frac{e^2}{r}+\frac{(l+1)(l+2)}{r^2}+\frac{e^4}{2(l+1)^2}-\frac{e^4}{4(l+2)^2}+\left(\frac{1}{4}-\mu^2\right)\frac{1}{r^2}+\frac{1}{2}(\mu+1)-\frac{r^2}{16},\\
E_n^{(2)} &= \frac{e^4}{4(l+2)^2}-\frac{e^4}{4(n+l+2)^2},\\
\psi_n^{(2)}(r) &= \frac{n+1}{2(l+1)(n+l+2)^2}r^{\frac{1}{2}-\mu}\exp\left\{\frac{r^2}{8}-\frac{y(r;l+1)}{2}\right\}y(r;l+1)^l\\
&\quad\times\left\{L_n^{2l+1}(y(r;l+1))-\frac{2(l+1)}{n+1}L_{n+1}^{2l+2}(y(x;l+1))\right\}.
\end{aligned}
\tag{100}
$$

5. Conclusions

The one-dimensional problems are useful for the investigation of solvable potential problems. Multivariable problems are often analytically solvable when the conditions of variable separability are fulfilled and the corresponding uncoupled differential equations generated by this separation are algebraically solvable. The so-obtained one-variable problems can be expressed in terms of SL problems (70). But, as the SL problems are not generally analytically solvable, one used to transform them into the Schrödinger-type problems

$$H\psi = E\psi, \qquad H = -D^2+U, \tag{101}$$

Table 3. Solvable potentials for Laguerre SL-type problem.

Type of potentials	Variable y	Potential V_1	Eigenvalues $E_n^{(1)}$	Eigenfunctions $\psi_n^{(1)}$
Three-dimensional oscillator LI	$\frac{1}{2}\omega r^2$	$\frac{1}{4}\omega^2 r^2 + \frac{l(l+1)}{r^2} - \left(l + \frac{3}{2}\right)\omega$ $+\left(\frac{1}{4} - \mu^2\right)\frac{1}{r^2} + \frac{1}{2}(\mu+1) - \frac{r^2}{16}$	$2n\omega$	$r^{-\mu-\frac{1}{2}}\,\mathrm{e}^{\left(\frac{r^2}{8}-\frac{y}{2}\right)}$ $\times\, y^{\frac{l+1}{2}} L_n^{l+\frac{1}{2}}(y)$
Coulomb LII	$\frac{re^2}{(n+l+1)}$	$-\frac{e^2}{r} + \frac{l(l+1)}{r^2} + \frac{e^4}{4(l+1)^2}$ $+\left(\frac{1}{4} - \mu^2\right)\frac{1}{r^2} + \frac{1}{2}(\mu+1) - \frac{r^2}{16}$	$\frac{e^4}{4(l+1)^2}$ $-\frac{e^4}{4(n+l+1)^2}$	$r^{-\mu-\frac{1}{2}}\,\mathrm{e}^{\left(\frac{r^2}{8}-\frac{y}{2}\right)} y^{l+1}$ $\times L_n^{2l+1}(y)$
Rosen–Morse	$\cosh(ar)$	$(B^2 + A^2 + Aa)\,\mathrm{cosech}^2(ar)$ $- B(2A+a)\coth(ar)\,\mathrm{cosech}(ar)$ $+ A^2 + \left(\frac{1}{4} - \mu^2\right)\frac{1}{r^2}$ $+ \frac{1}{2}(\mu+1) - \frac{r^2}{16}$	$A^2 - (A - na)^2$	$r^{-\mu-\frac{1}{2}}\,\mathrm{e}^{\frac{r^2}{8}} \cosh^{-\lambda-s}\left(\frac{ar}{2}\right)$ $\times \sinh^{\lambda-s}\left(\frac{ar}{2}\right)$ $\times P_n^{\left(\lambda-s-\frac{1}{2},-\lambda-s-\frac{1}{2}\right)}(y)$
Eckart PII	$\coth(ar)$ $\bar{a} = \frac{\lambda}{(n+s)}$	$A^2 + \frac{B^2}{A^2} - 2B\coth(ar)$ $+ A(A-a)\,\mathrm{cosech}^2(ar)$ $+\left(\frac{1}{4} - \mu^2\right)\frac{1}{r^2} + \frac{1}{2}(\mu+1) - \frac{r^2}{16}$	$A^2 - (A + na)^2$ $+ \frac{B^2}{A^2} - \frac{B^2}{(A+na)^2}$	$r^{-\mu-\frac{1}{2}}\,\mathrm{e}^{\frac{r^2}{8} - \frac{Br}{A+an}}$ $\times \sinh^{n+s}(ar)$ $\times P_n^{(-s-n+\bar{a},-s-n-\bar{a})}(y)$
Eckart PI	$\cos(a\theta)$	$-A^2 + (B^2 + A^2 - Aa)\csc^2(a\theta)$ $- B(2A - a)\cot(a\theta)\csc(a\theta)$ $+\left(\frac{1}{4} - \mu^2\right)\frac{1}{\theta^2} + \frac{1}{2}(\mu+1) - \frac{\theta^2}{16}$	$(A + na)^2 - A^2$	$\theta^{-\mu-\frac{1}{2}}\,\mathrm{e}^{\frac{\theta^2}{8}} \cos^{s+\lambda}\left(\frac{a\theta}{2}\right)$ $\times \sin^{s-\lambda}\left(\frac{a\theta}{2}\right)$ $\times P_n^{\left(s-\lambda-\frac{1}{2},s+\lambda-\frac{1}{2}\right)}(y)$
Poschl–Teller I PI	$\cos(2a\delta)$	$-(A+B)^2 + A(A-a)\sec^2(a\delta)$ $+ B(B-a)\csc^2(a\delta)$ $+\left(\frac{1}{4} - \mu^2\right)\frac{1}{\delta^2} + \frac{1}{2}(\mu+1) - \frac{\delta^2}{16}$	$(A + B + 2na)^2$ $- (A+B)^2$	$\delta^{-\mu-\frac{1}{2}}\,\mathrm{e}^{\frac{\delta^2}{8}} \cos^s(a\delta)$ $\times \sin^\lambda(a\delta)$ $\times P_n^{\left(\lambda-\frac{1}{2},s-\frac{1}{2}\right)}(y)$
Poschl–Teller II PI	$\cosh(2ar)$	$(A-B)^2 - A(A+a)\,\mathrm{sech}^2(ar)$ $-B(B-a)\,\mathrm{cosech}^2(ar)$ $+\left(\frac{1}{4} - \mu^2\right)\frac{1}{\theta^2} + \frac{1}{2}(\mu+1) - \frac{\theta^2}{16}$	$(A-B)^2$ $-(A + B - 2na)^2$	$r^{-\mu-\frac{1}{2}}\,\mathrm{e}^{\frac{r^2}{8}} \cosh^{-s}(ar)$ $\times \sinh^\lambda(ar)$ $\times P_n^{\left(\lambda-\frac{1}{2},-s-\frac{1}{2}\right)}(y)$
Lévai PII	$-\mathrm{i}\cot(a\theta)$ $\bar{a} = \frac{\lambda}{(s-n)}$	$A(A+a)\csc^2(a\theta)$ $- 2B\cot(a\theta) - A^2 + \frac{B^2}{A^2}$ $+\left(\frac{1}{4} - \mu^2\right)\frac{1}{\theta^2} + \frac{1}{2}(\mu+1) - \frac{\theta^2}{16}$	$-A^2 + (A - na)^2$ $+ \frac{B^2}{A^2} - \frac{B^2}{(A-na)^2}$	$\mathrm{e}^{\left(\frac{\theta^2}{8} + \frac{B\theta}{A-an}\right)}\theta^{-\mu-\frac{1}{2}}$ $\times \sin^{-s+n}(a\theta)$ $\times P_n^{(s-n+\mathrm{i}\bar{a},s-n-\mathrm{i}\bar{a})}(y)$

where U is the corresponding effective potential function. Indeed, up until today, to our best knowledge of the literature, almost all the works on exactly solvable problems are based on equation (101). In this respect, it is well known that the concepts of SUSY QM factorization and shape invariance are very powerful tools for solving problems of form (101). The more simple the effective potential is, the easier it is to solve the Riccati equation which arises from the factorization method. Unfortunately, due to the presence of an extrapotential term which arises when one transforms a SL problem (70) into a Schrödinger problem (101), the resulting effective potential does not usually factorize, i.e., the corresponding Riccati-type equation cannot always be analytically solved. In the case when the resulting potential factorizes, it does not often fulfil the suitable shape invariance condition. Moreover, the extrapotential term restricts the possibility of *fitting* the resulting potential with a known solvable potential. Hence, transforming a SL equation into a Schrödinger one does not always lead to exactly solvable potentials. This justifies the necessity of searching for a direct method for analytically solving SL problems. These series of works in progress aim at providing such a tool of constructing the SL solvable potentials.

Table 4. Shape invariance data on potentials for Laguerre SL-type problem.

Type of potentials	Variable y	Superpotential W	a_1	a_2	$\mathcal{R}(a_1)$
Three-dimensional oscillator LI	$\frac{1}{2}\omega r^2$	$\frac{\omega}{2}r - \frac{(l+1)}{r} - \frac{r}{4} + \frac{1+2\mu}{2r}$	l	$l+1$	2ω
Coulomb LII	$\frac{re^2}{(n+l+1)}$	$-\frac{(l+1)}{r} - \frac{r}{4} + \frac{1+2\mu}{2r} + \frac{e^2}{2(l+1)}$	l	$l+1$	$\frac{e^4}{4(l+1)^2} - \frac{e^4}{4(l+2)^2}$
Rosen–Morse PI	$\cosh(ar)$	$A\coth(ar) - B\,\mathrm{cosech}(ar)$ $- \frac{r}{4} + \frac{1+2\mu}{2r}$	A	$A-a$	$A^2 - (A-a)^2$
Eckart PII	$\coth(ar)$	$-A\coth(ar) + \frac{B}{A}$ $- \frac{r}{4} + \frac{1+2\mu}{2r}$	A	$A+a$	$A^2 - (A+a)^2$ $+ \frac{B^2}{A^2} - \frac{B^2}{(A-a)^2}$
Eckart PI	$\cos(a\theta)$	$-A\cot(a\theta) + B\csc(a\theta)$ $- \frac{\theta}{4} + \frac{1+2\mu}{2\theta}$	A	$A+a$	$(A+a)^2 - A^2$
Pöschl–Teller I PI	$\cos(2a\delta)$	$A\tan(a\delta) - B\cot(a\delta)$ $- \frac{\delta}{4} + \frac{1+2\mu}{2\delta}$	(A, B)	$(A+a, B+a)$	$-(A+B)^2$ $(A+B+2a)^2$
Pöschl–Teller II PI	$\cosh(2ar)$	$A\tanh(ar) - B\coth(ar)$ $- \frac{r}{4} + \frac{1+2\mu}{2r}$	(A, B)	$(A-a, B+a)$	$(A-B)^2$ $-(A-B-2a)^2$
Lévai PII	$-\mathrm{i}\cot(a\theta)$	$-\frac{B}{A} + A\cot(a\theta) - \frac{\theta}{4} + \frac{1+2\mu}{2\theta}$	A	$A-a$	$-A^2 + (A-a)^2$ $+ \frac{B^2}{A^2} - \frac{B^2}{(A-a)^2}$

Table 5. Solvable potentials for Legendre SL-type problem.

Type of potentials	Variable y	Potential V_1	Eigenvalues $E_n^{(1)}$	Eigenfunctions $\psi_n^{(1)}$
Eckart PI	$\cos(a\theta)$	$(B^2 + A^2 - Aa)\csc^2(a\theta)$ $- B(2A-a)\cot(a\theta)\csc(a\theta)$ $- A^2 - \frac{1}{4}\left(1 - \frac{1}{2}\csc^2\theta\right)$	$(A+na)^2 - A^2$	$1/\sqrt{\sin\theta}\cos^{s+\lambda}\left(\frac{a\theta}{2}\right)$ $\times \sin^{s-\lambda}\left(\frac{a\theta}{2}\right)$ $\times P_n^{(s-\lambda-\frac{1}{2}, s+\lambda-\frac{1}{2})}(y)$
Poschl–Teller I PI	$\cos(2a\delta)$	$A(A-a)\sec^2(a\delta)$ $+ B(B-a)\csc^2(a\delta)$ $-(A+B)^2 - \frac{1}{4}\left(1 - \frac{1}{2}\csc^2\delta\right)$	$(A+B+2na)^2$ $-(A+B)^2$	$1/\sqrt{\sin\delta}\cos^s(a\delta)$ $\times \sin^\lambda(a\delta)$ $\times P_n^{(\lambda-\frac{1}{2}, s-\frac{1}{2})}(y)$
Lévai PII	$-\mathrm{i}\cot(a\theta)$ $\bar{a} = \frac{\lambda}{(s-n)}$	$A(A+a)\csc^2(a\theta)$ $-2B\cot(a\theta) - A^2 + \frac{B^2}{A^2}$ $- \frac{1}{4}\left(1 - \frac{1}{2}\csc^2\theta\right)$	$-A^2 + (A-na)^2$ $+ \frac{B^2}{A^2} - \frac{B^2}{(A-na)^2}$	$1/\sqrt{\sin\theta}\,\mathrm{e}^{\frac{B\theta}{A-an}}\sin^{n-s}(a\theta)$ $\times P_n^{(s-n+\mathrm{i}\bar{a}, s-n-\mathrm{i}\bar{a})}(y)$

Table 6. Shape invariance data on potentials for Legendre SL-type problem.

Type of potentials	Variable y	Superpotential W	a_1	a_2	$\mathcal{R}(a_1)$
Eckart PI	$\cos(a\theta)$	$-A\cot(a\theta) - B\csc(a\theta) + \cot\left(\frac{\theta}{2}\right)$	A	$A+a$	$(A+a)^2 - A^2$
Pöschl–Teller I PI	$\cos(2a\delta)$	$A\tan(a\delta) - B\cot(a\delta) + \cot\left(\frac{\delta}{2}\right)$	(A, B)	$(A+a, B+a)$	$(A+B+2a)^2$ $-(A+B)^2$
Lévai PII	$-\mathrm{i}\cot(a\theta)$	$-\frac{B}{A} + A\cot(a\theta) + \cot\left(\frac{\theta}{2}\right)$	A	$A-a$	$-A^2 + (A-a)^2$ $+ \frac{B^2}{A^2} - \frac{B^2}{(A-a)^2}$

In this paper, we have extended the formulation of SUSY QM to SL operators. Namely, we have provided a general method of factorization of these operators. The factorizing operators fulfil a superalgebra. We have shown that the transformed operators B and $B^\dagger$ associated with

Table 7. Shape invariance algebras.

Type of potentials	$\mathcal{R}(a_0)$	Commutation relations	Type of algebra
Shifted oscillator HI	ω	$[B_-, B_+] = \mathcal{R}(a_0)$ $[B_\pm, \mathcal{R}(a_0)] = 0$	Weyl–Heisenberg
Three-dimensional oscillator LI	2ω	$[B_-, B_+] = \mathcal{R}(a_0)$ $[B_\pm, \mathcal{R}(a_0)] = 0$	Weyl–Heisenberg
Coulomb LII	$\frac{e^4}{4l^2} - \frac{e^4}{4(l+1)^2}$	$[B_-, B_+] = \mathcal{R}(a_0)$	Infinite dimensional
Morse LIII	$a^2 + 2aA$	$[B_-, B_+] = \mathcal{R}(a_0)$ $[B_\pm, \mathcal{R}(a_0)] = -2a^2 B_\pm$	$su(1, 1)$
Morse PI	$a^2 + 2aA$	$[B_-, B_+] = \mathcal{R}(a_0)$ $[B_\pm, \mathcal{R}(a_0)] = -2a^2 B_\pm$	$su(1, 1)$
Rosen–Morse PII	$(A + a)^2 - A^2 + \frac{B^2}{(A+a)^2} - \frac{B^2}{A^2}$	$[B_-, B_+] = \mathcal{R}(a_0)$	Infinite dimensional
Rosen–Morse PI	$a^2 + 2aA$	$[B_-, B_+] = \mathcal{R}(a_0)$ $[B_\pm, \mathcal{R}(a_0)] = -2a^2 B_\pm$	$su(1, 1)$
Eckart PII	$(A - a) - A^2 + \frac{B^2}{(A-a)^2} - \frac{B^2}{A^2}$	$[B_-, B_+] = \mathcal{R}(a_0)$	Infinite dimensional
Eckart PI	$-a^2 + 2aA$	$[B_-, B_+] = \mathcal{R}(a_0)$ $[B_\pm, \mathcal{R}(a_0)] = +2a^2 B_\pm$	$su(2)$
Pöschl–Teller I PI	$-4a^2 + 4a(A + B)$	$[B_-, B_+] = \mathcal{R}(a_0)$ $[B_\pm, \mathcal{R}(a_0)] = 8a^2 B_\pm$	$su(2)$
Pöschl–Teller II PI	$-4a^2 + 4a(A - B)$	$[B_-, B_+] = \mathcal{R}(a_0)$ $[B_\pm, \mathcal{R}(a_0)] = -8a^2 B_\pm$	$su(1, 1)$
Lévai PII	$A^2 - (A + a)^2 + \frac{B^2}{(A+a)^2} - \frac{B^2}{A^2}$	$[B_-, B_+] = \mathcal{R}(a_0)$	Infinite dimensional

SL shape invariant potentials form an algebra which is in general infinite-dimensional. The condition these operators fulfil to define a finite algebra has been explicitly deduced. We have given two examples which show how to construct a new SL solvable potential from an old one. Finally, we have extended the Lévai method of constructing exactly solvable potentials to SL problems. Twenty-three SL shape invariant potentials have been obtained. Twelve of them are for Hermite SL-type problem, eight for Laguerre SL-type problem and three for Legendre SL-type problem. We have also given the algebras associated with the operators of these shape invariant potentials. Some algebras are infinite dimensional, while the others which are finite are isomorphic to Heisenberg–Weyl, $su(2)$ or $su(1, 1)$ algebras.

The potentials we obtained differ from those of Dabrowska *et al* [18] and of Lévai [22] by the presence of the extrapotential terms. The method we develop thus allows the investigation of exactly solvable potentials associated with SL operators as well as it shows the way of obtaining new exactly solvable potentials from a given old one. Besides, provided an exactly solvable potential V_{Sch} for the Schrödinger operator, our method furnishes the corresponding SL potential V_{SL} as follows: $V_{\mathrm{SL}} = V_{\mathrm{ex}} + V_{\mathrm{Sch}}$ where V_{ex}, the extrapotential component, is expressed in terms of SL parameters σ and τ (83). Conversely, from a given SL exactly solvable potential we can retrieve the corresponding Schrödinger exactly solvable potential using equations (48)–(50).

In the forthcoming papers, we investigate the SL shape invariant potentials of the second class for which the parameters a_1 and a_2 are related by scaling, SL self-similar potentials and SL quasi-exactly solvable potentials.

Acknowledgments

The authors are very thankful to the referees for their remarks that help them to improve the conclusion section of the paper. They are also grateful to the Belgian Cooperation CUD-CIUF-UAC/IMSP and the Centre Béninois de la Recherche Scientifique et Technique (CBRST) for their financial support.

References

[1] Morse P M 1929 *Phys. Rev.* **34** 57
[2] Eckart C 1930 *Phys. Rev.* **35** 1303
[3] Pöschl G and Teller E 1933 *Z. Phys.* **83** 143
[4] Rosen N and Morse P 1932 *Phys. Rev.* **42** 210
[5] Gionocchio J N 1984 *Ann. Phys., NY* **152** 203
[6] Cooper F, Gionocchio J N and Khare A 1987 *Phys. Rev.* D **36** 3458
[7] Dutt R, Khare A and Sukhatme U P 1986 *Phys. Lett.* B **181** 295
[8] Dutt R, Khare A and Sukhatme U P 1988 *Am. J. Phys.* **56** 163
[9] Baloïtcha E and Hounkonnou M N 1999 *J. Math. Phys.* **40** 6133
[10] Mielnik B 1984 *J. Math. Phys.* **40** 3387
[11] Bargrov V G and Samsonov B F 1995 *Theor. Math. Phys.* **104** 1051
[12] Fernandez D J, Hussin V and Nieto L M 1994 *J. Phys. A: Math. Gen.* **27** 3547
[13] Schrödinger E 1940 *Proc. R. Irish Acad.* A **46** 9
Schrödinger E 1941 *Proc. R. Irish Acad.* A **46** 183
Schrödinger E 1941 *Proc. R. Irish Acad.* A **47** 53
[14] Infeld L and Hull T E 1951 *Rev. Mod. Phys.* **23** 28
[15] Witten E 1981 *Nucl. Phys.* B **185** 5123
[16] Gendenshtein L 1983 *JETP Lett.* **38** 356
[17] Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* **251** 267
[18] Dabrowska J, Khare A and Sukhatme U 1988 *J. Phys. A: Math. Gen.* **21** L195
[19] Khare A and Sukhatme U 1991 *J. Phys. A: Math. Gen.* **26** L901
[20] Fukui T and Aizawa N 1993 *Phys. Lett.* A **180** 308
[21] Balantekin A B 1998 *Phys. Rev.* A **57** 4188
[22] Levai G 1989 *J. Phys. A: Math. Gen.* **22** 689
[23] Levai G 1991 *J. Phys. A: Math. Gen.* **24** 131
Levai G 1992 *J. Phys. A: Math. Gen.* **25** L521
[24] Jafarizadeh M A and Fakhri H 1997 *Phys. Lett.* A **230** 157
Jafarizadeh M A and Fakhri H 1997 *Phys. Lett.* A **230** 164
[25] Jafarizadeh M A and Fakhri H 1998 *Ann. Phys., NY* **262** 260
[26] Lorente M 2001 *J. Phys. A: Math. Gen* **34** 569
[27] Cotfas N 2002 *J. Phys. A: Math. Gen.* **35** 9355
[28] Hounkonnou M N, Sodoga K, Justum Y and Lauvergnat D 2004 *J. Phys. B: At. Mol. Opt. Phys.* **37** 1859
[29] Byron F W Jr and Fuller R W 1992 *Mathematics of Classical and Quantum Physics* (New York: Dover)
[30] Teschl G 1999 Schrödinger operators course Online at http://www.mat.univie.ac.at/~gerald
[31] Dautray R and Lions J-L 2000 *Mathematical Analysis and Numerical Methods for Science and Technology: Spectral Theory and Applications* (Berlin: Springer)
[32] Nikiforov A F, Suslov S K and Uvarov V B 1991 *Classical Orthogonal Polynomials of a Discrete Variable* (Berlin: Springer)
[33] Abramowitz M and Stegun J A (ed) 1965 *Handbook of Mathematical Functions* (New York: Dover)